# SyDe312 (Winter 2005) Unit 4 - Solutions

## Problem 5.1 - 2

a) Using the program provided in the text in page 194
We got the following table:

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 2.6516335857D+01 | -2.52D+01 | |
| 4 | 3.2490504945D+00 | -1.95D+00 | 1.30D+01 |
| 8 | 1.6245252472D+00 | -3.22D-01 | 6.04D+00 |
| 16 | 1.3757225177D+00 | -7.33D-02 | 4.39D+00 |
| 32 | 1.3203118784D+00 | -1.79D-02 | 4.09D+00 |
| 64 | 1.3068478855D+00 | -4.45D-03 | 4.02D+00 |
| 128 | 1.3035056585D+00 | -1.11D-03 | 4.01D+00 |
| 256 | 1.3026715795D+00 | -2.78D-04 | 4.00D+00 |
| 512 | 1.3024631519D+00 | -6.95D-05 | 4.00D+00 |

d)

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 9.6954615725D+00 | -1.74D+00 | |
| 4 | 7.9893234398D+00 | -3.44D-02 | 5.06D+01 |
| 8 | 7.9549277727D+00 | -1.25D-06 | 2.75D+04 |
| 16 | 7.9549265210D+00 | -4.26D-14 | 2.94D+07 |
| 32 | 7.9549265210D+00 | -4.00D-14 | 1.07D+00 |
| 64 | 7.9549265210D+00 | -4.00D-14 | 1.00D+00 |
| 128 | 7.9549265210D+00 | -4.71D-14 | 8.49D-01 |
| 256 | 7.9549265210D+00 | -4.44D-14 | 1.06D+00 |
| 512 | 7.9549265210D+00 | -4.97D-14 | 8.93D-01 |

Here we can notice the effect of the periodic integrand.

e)

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 1.9211791672D+00 | 1.86D-02 | |
| 4 | 1.9351196587D+00 | 4.62D-03 | 4.02D+00 |
| 8 | 1.9385825146D+00 | 1.15D-03 | 4.01D+00 |
| 16 | 1.9394468577D+00 | 2.88D-04 | 4.00D+00 |
| 32 | 1.9394468577D+00 | 7.20D-05 | 4.00D+00 |
| 64 | 1.9396628581D+00 | 1.80D-05 | 4.00D+00 |
| 128 | 1.9397168528D+00 | 4.50D-06 | 4.00D+00 |
| 256 | 1.9397303512D+00 | 1.12D-06 | 4.00D+00 |
| 512 | 1.9397345694D+00 | 2.81D-07 | 4.00D+00 |

f)

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 6.0355339059D-01 | 6.31D-02 | |
| 4 | 6.4328304624D-01 | 2.34D-02 | 2.70D+00 |
| 8 | 6.5813022162D-01 | 8.54D-03 | 2.74D+00 |
| 16 | 6.6358119688D-01 | 3.09D-03 | 2.77D+00 |
| 32 | 6.6555893628D-01 | 1.11D-03 | 2.79D+00 |
| 64 | 6.6627081138D-01 | 3.96D-04 | 2.80D+00 |
| 128 | 6.6652565730D-01 | 1.41D-04 | 2.81D+00 |
| 256 | 6.6661654898D-01 | 5.01D-05 | 2.81D+00 |
| 512 | 6.6664888155D-01 | 1.78D-05 | 2.82D+00 |

## Problem 5.1 - 3

a) Using the program (Simpson's rule)provided in the text in page 198
We got the following tabel:

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 2.2715077371D+01 | -2.14D+01 | |
| 4 | -4.5067112930D+00 | 5.81D+00 | -3.69D+00 |
| 8 | 1.0830168315D+00 | 2.19D-01 | 2.65D+01 |
| 16 | 1.2927882745D+00 | 9.61D-03 | 2.28D+01 |
| 32 | 1.3018416653D+00 | 5.52D-04 | 1.74D+01 |
| 64 | 1.3023598879D+00 | 3.38D-05 | 1.63D+01 |
| 128 | 1.3023915828D+00 | 2.10D-06 | 1.61D+01 |
| 256 | 1.3023935531D+00 | 1.31D-07 | 1.60D+01 |
| 512 | 1.3023936761D+00 | 8.20D-09 | 1.60D+01 |

d)

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 1.2156797197D+01 | -4.20D+00 | |
| 4 | 7.4206107289D+00 | 5.34D-01 | -7.86D+00 |
| 8 | 7.9434625503D+00 | 1.15D-02 | 4.66D+01 |
| 16 | 7.9549261038D+00 | 4.17D-07 | 2.75D+04 |
| 32 | 7.9549265210D+00 | -4.09D-14 | -1.02D+07 |
| 64 | 7.9549265210D+00 | -4.09D-14 | 1.00D+00 |
| 128 | 7.9549265210D+00 | -4.44D-14 | 9.20D-01 |
| 256 | 7.9549265210D+00 | -4.35D-14 | 1.02D+00 |
| 512 | 7.9549265210D+00 | -4.62D-14 | 9.42D-01 |

e)

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 1.9402703334D+00 | -5.35D-04 | |
| 4 | 1.9397664891D+00 | -3.16D-05 | 1.69D+01 |
| 8 | 1.9397368000D+00 | -1.95D-06 | 1.62D+01 |
| 16 | 1.9397349720D+00 | -1.21D-07 | 1.61D+01 |
| 32 | 1.9397348582D+00 | -7.58D-09 | 1.60D+01 |
| 64 | 1.9397348511D+00 | -4.73D-10 | 1.60D+01 |
| 128 | 1.9397358507D+00 | -2.93D-11 | 1.62D+01 |
| 256 | 1.9397348506D+00 | -1.50D-12 | 1.95D+01 |
| 512 | 1.9397348506D+00 | 2.36D-13 | -6.34D+00 |

f)

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 6.3807118746D-01 | 2.86D-02 | |
| 4 | 6.5652626479D-01 | 1.01D-02 | 2.82D+00 |
| 8 | 6.6307928009D-01 | 3.59D-03 | 2.83D+00 |
| 16 | 6.6539818863D-01 | 1.27D-03 | 2.83D+00 |
| 32 | 6.6621818275D-01 | 4.48D-04 | 2.83D+00 |
| 64 | 6.6650810308D-01 | 1.59D-04 | 2.83D+00 |
| 128 | 6.6661060594D-01 | 5.61D-05 | 2.83D+00 |
| 256 | 6.6664684620D-01 | 1.98D-05 | 2.83D+00 |
| 512 | 6.6665965907D-01 | 7.01D-06 | 2.83D+00 |

## Problem 5.1 - 4

b) Using the same program for trapezoidal rule that we used in problem 2, we get:

| $n$ | $T_n$ |
|---|---|
| 4 | 2.1774504814D+00 |
| 8 | 2.1750613565D+00 |
| 16 | 2.1744613309D+00 |
| 32 | 2.1743111490D+00 |
| 64 | 2.1742735926D+00 |
| 128 | 2.1742642028D+00 |
| 256 | 2.1742618553D+00 |
| 512 | 2.1742612684D+00 |

Using the same program for simpson's rule that we used in problem 3, we get:

| $n$ | $S_n$ |
|---|---|
| 4 | 2.1744179142D+00 |
| 8 | 2.1742649815D+00 |
| 16 | 2.1742613224D+00 |
| 32 | 2.1742610884D+00 |
| 64 | 2.1742610737D+00 |
| 128 | 2.1742610728D+00 |
| 256 | 2.1742610728D+00 |
| 512 | 2.1742610728D+00 |

## Problem 5.1 - 5

a) Using the same program for trapezoidal rule that we used in problem 2, we get:

| $n$ | $T_n$ |
|---|---|
| 4 | 2.2922995506D+00 |
| 8 | 2.3045646493D+00 |
| 16 | 2.3048919200D+00 |
| 32 | 2.3048926613D+00 |
| 64 | 2.3048926613D+00 |
| 128 | 2.3048926613D+00 |
| 256 | 2.3048926613D+00 |
| 512 | 2.3048926613D+00 |

Using the same program for simpson's rule that we used in problem 3, we get:

| $n$ | $S_n$ |
|---|---|
| 4 | 2.3402480159D+00 |
| 8 | 2.3086530155D+00 |
| 16 | 2.3050010103D+00 |
| 32 | 2.3048929084D+00 |
| 64 | 2.3048926614D+00 |
| 128 | 2.3048926614D+00 |
| 256 | 2.3048926614D+00 |
| 512 | 2.3048926614D+00 |

## Problem 5.1 - 6

Using the program Trapezoidal rule when $b = \frac{\pi}{2}$

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 6.1737379461D-01 | -1.27D-02 | |
| 4 | 6.0780823007D-01 | 3.20D-03 | 3.98D+00 |
| 8 | 6.0540271743D-01 | 8.03D-04 | 4.00D+00 |
| 16 | 6.0480056952D-01 | 2.01D-04 | 4.00D+00 |
| 32 | 6.0464998647D-01 | 5.02D-05 | 4.00D+00 |
| 64 | 6.0461233787D-01 | 1.25D-05 | 4.00D+00 |
| 128 | 6.0460292554D-01 | 3.14D-06 | 4.00D+00 |
| 256 | 6.0460057244D-01 | 7.84D-07 | 4.00D+00 |
| 512 | 6.0459998417D-01 | 1.96D-07 | 4.00D+00 |

Using the simpson's rule:

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 6.0499890299D-01 | 3.99D-01 | |
| 4 | 6.0461970856D-01 | 1.99D-02 | 2.00D+01 |
| 8 | 6.0460087989D-01 | 1.09D-03 | 1.82D+01 |
| 16 | 6.0459985354D-01 | 6.54D-05 | 1.67D+01 |
| 32 | 6.0459979212D-01 | 4.04D-06 | 1.62D+01 |
| 64 | 6.0459978833D-01 | 2.52D-07 | 1.60D+01 |
| 128 | 6.0459978809D-01 | 1.58D-08 | 1.60D+01 |
| 256 | 6.0459978808D-01 | 9.86D-10 | 1.60D+01 |
| 512 | 6.0459978808D-01 | 6.40D-11 | 1.54D+01 |

Using the program Trapezoidal rule when $b = \pi$

| $n$ | $T_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 1.3089969390D+00 | 9.98D-02 | |
| 4 | 1.2347475892D+00 | 2.55D-02 | 3.91D+00 |
| 8 | 1.2156164602D+00 | 6.42D-03 | 3.98D+00 |
| 16 | 1.2108054349D+00 | 1.61D-03 | 4.00D+00 |
| 32 | 1.2096011390D+00 | 4.02D-04 | 4.00D+00 |
| 64 | 1.2092999729D+00 | 1.00D-04 | 4.00D+00 |
| 128 | 1.2092246757D+00 | 2.51D-05 | 4.00D+00 |
| 256 | 1.2092058511D+00 | 6.27D-06 | 4.00D+00 |
| 512 | 1.2092011449D+00 | 1.57D-06 | 4.00D+00 |

Using the simpson's rule:

| $n$ | $S_n$ | Error | Ratio |
|---|---|---|---|
| 2 | 1.2217304763D+00 | 1.25D-02 | |
| 4 | 1.2099978059D+00 | 7.98D-04 | 1.57D+01 |
| 8 | 1.2092394171D+00 | 3.98D-05 | 2.00D+01 |
| 16 | 1.2092017597D+00 | 2.18D-06 | 1.82D+01 |
| 32 | 1.2091997070D+00 | 1.30D-07 | 1.67D+01 |
| 64 | 1.2091995842D+00 | 8.09D-09 | 1.62D+01 |
| 128 | 1.2091995766D+00 | 5.04D-10 | 1.60D+01 |
| 256 | 1.2091995761D+00 | 3.15D-11 | 1.60D+01 |
| 512 | 1.2091995761D+00 | 1.97D-12 | 1.60D+01 |

## Problem 5.3 - 1

Using the following formula:
$I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$ The values for each x and w for specific n's are taken from table 5.7 in page 223 in the text
$I_3 = 0.5555555556f(-0.7745966692)+0.8888888889f(0)+0.5555555556f(0.7745966692) = 2.3503369287$
We know that $I = 2.3504024$
Then, $I - I_3 = 6.55E-05$

Likewise,
$I_n = 0.347854851f(-0.8611363116)+0.6521451549f(-0.3399810436)+0.6521451549f(0.3399810436)+0.347854851f(0.8611363116) = 2.3504020922$
Then, $I - I_4 = 3.08E-07$

## Problem 5.3 - 2

a) Using the same procedure in problem 1 after coverting to the standard interval [-1,1]
$x = \frac{b+a+t(b-a)}{2}$ for the interval $-1 \le t \le 1$
This transform the integral to:
$I(f) = \frac{b-a}{2}\int_1^{-1} \widetilde{f}(t)dt$
with $\widetilde{f}(t) = f(\frac{b+1+t(b-a)}{2})$
We got the following results:

| $n$ | $I_n$ | Error |
|---|---|---|
| 2 | -1.9244871326D+01 | 2.05D+01 |
| 3 | 9.0897287084D+00 | -7.79D+00 |
| 4 | 9.2205252152D-01 | 3.80D-01 |

d)

| $n$ | $I_n$ | Error |
|---|---|---|
| 2 | 4.9394725384 | 3.02D+00 |
| 3 | 9.2240204504 | -1.27D+00 |
| 4 | 7.5169093090 | 4.38D-01 |

e)

| $n$ | $I_n$ | Error |
|---|---|---|
| 2 | 1.9393761294 | 3.59D-04 |
| 3 | 1.9397367251 | -1.87D-06 |
| 4 | 1.9397348494 | 1.26D-09 |

f)

| $n$ | $I_n$ | Error |
|---|---|---|
| 2 | 1.3477746774 | -0.68D+00 |
| 3 | 1.3383592679 | -0.67D+00 |
| 4 | 1.3356552979 | -0.66D+00 |

## Problem 5.3 - 4

b) $I(f) = \int_0^2 tan^{-1}(1+x^2)dx$
To transform the integrals to the standard interval [-1,1], we introduce the linear change of variable
$x = \frac{b+a+t(b-a)}{2} = 1+t$ for the interval $-1 \le t \le 1$
The integral is transformed to:
$I(f) = \int_{-1}^{1} \widetilde{f}(t)dt$
with $\widetilde{f}(t) = f(1+t) = tan^{-1}(1+(1+t)^2)$
Then, using the gaussian integration program:

| $n$ | $I_n$ |
|---|---|
| 1 | 2.2142974356D+00 |
| 2 | 2.1587989303D+00 |
| 3 | 2.1755005241D+00 |
| 4 | 2.1742436477D+00 |
| 5 | 2.1742526878D+00 |
| 6 | 2.1742626644D+00 |
| 7 | 2.1742608924D+00 |
| 8 | 2.1742610829D+00 |

## Problem 5.3 - 5

a) $I(f) = \int_0^1 \sqrt{1+\pi^2 cos^2(\pi x)}dx$
To transform the integrals to the standard interval [-1,1], we introduce the linear change of variable
$x = \frac{b+a+t(b-a)}{2} = \frac{1+t}{2}$ for the interval $-1 \le t \le 1$
The integral is transformed to:
$I(f) = \frac{1}{2}\int_{-1}^{1} \widetilde{f}(t)dt$
with $\widetilde{f}(t) = f(\frac{1+t}{2}) = \sqrt{1+\pi^2 cos^2(\frac{\pi(1+t)}{2})}$

Then, using the gaussian integration program:

| $n$ | $I_n$ |
|---|---|
| 1 | 1.0000000000D+00 |
| 2 | 2.6687463303D+00 |
| 3 | 2.1732125861D+00 |
| 4 | 2.3521766514D+00 |
| 5 | 2.2809438453D+00 |
| 6 | 2.3162359893D+00 |
| 7 | 2.2986620428D+00 |
| 8 | 2.3082020963D+00 |

## Problem 5.4 - 1

a) $f(x) = e^x$ at $x = 0$
Using the follwoing equations:
$\acute{f}(x) \approx \frac{f(x+h)-f(x)}{h} \equiv D_h f(x)$
$\acute{f}(x) - D_h(x) = -\frac{h}{2}\acute{\acute{f}}(c)$

We get the following results:

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | 1.051709 | -5.17E-2 | | -5.00E-2 |
| 0.05 | 1.025442 | -2.54E-2 | 2.03 | -2.50E-2 |
| 0.025 | 1.012605 | -1.26E-2 | 2.02 | -1.25E-2 |
| 0.0125 | 1.006276 | -6.28E-3 | 2.01 | -6.25E-3 |
| 0.00625 | 1.003132 | -3.13E-3 | 2.00 | -3.12E-3 |

c) Similar to part a when $f(x) = tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | 3.40979 | -2.91E+0 | | -4.79E+0 |
| 0.05 | 2.59405 | -2.09E+0 | 1.39 | -2.49E+0 |
| 0.025 | 1.67567 | -1.18E+0 | 1.78 | -1.24E+0 |
| 0.0125 | 1.10933 | -6.09E-1 | 1.93 | -6.22E-1 |
| 0.00625 | 0.80839 | -3.08E-1 | 1.98 | -3.11E-1 |

## Problem 5.4 - 2

a) $f(x) = e^x$ at $x = 0$
Using the follwoing equations:
$\acute{f}(x) \approx \frac{f(x)-f(x-h)}{h}$ $h \succ 0$
$\acute{f}(x) - \frac{f(x)-f(x-h)}{h} = \frac{h}{2}\acute{\acute{f}}(c)$

We get the following results:

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | 0.951626 | 4.84E-2 | | 5.00E-2 |
| 0.05 | 0.975412 | 2.46E-2 | 1.97 | 2.50E-2 |
| 0.025 | 0.987604 | 1.24E-2 | 1.98 | 1.25E-2 |
| 0.0125 | 0.993776 | 6.23E-3 | 1.99 | 6.25E-3 |
| 0.00625 | 0.996882 | 3.12E-3 | 2.00 | 3.12E-3 |

c) Similar to part a when $f(x) = tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | -3.00920 | 3.51E+0 | | 4.79E+0 |
| 0.05 | -1.81320 | 2.31E+0 | 1.52 | 2.49E+0 |
| 0.025 | -0.73611 | 1.24E+0 | 1.87 | 1.24E+0 |
| 0.0125 | -0.12480 | 6.25E-1 | 1.98 | 6.22E-1 |
| 0.00625 | 0.18771 | 3.12E-1 | 2.00 | 3.11E-1 |

## Problem 5.4 - 3

a) $f(x) = e^x$ at $x = 0$
Using the follwoing equations:
$\acute{f}(x_1) \approx \frac{f(x_1+h)-f(x_1-h)}{2h} \equiv D_h f(x_1)$
$\acute{f}(x_1) - \frac{f(x_1+h)-f(x_1-h)}{2h} = \frac{h^2}{6}\acute{\acute{\acute{f}}}(c_2)$ with $x_1 - h \leq c_2 \leq x_1 + h$
We get the following results:

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | 1.001667500 | -1.67E-3 | | -1.67E-3 |
| 0.05 | 1.000416719 | -4.17E-4 | 4.00 | -4.17E-4 |
| 0.025 | 1.000104170 | -1.04E-4 | 4.00 | -1.04E-4 |
| 0.0125 | 1.000026042 | -2.60E-5 | 4.00 | -2.60E-5 |
| 0.00625 | 1.000006510 | -6.51E-6 | 4.00 | -6.51E-3 |

c) Similar to part a when $f(x) = tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$

| $h$ | $D_h f(x)$ | Error | Ratio | Estimate |
|---|---|---|---|---|
| 0.1 | 0.200294 | 3.00E-1 | | 4.99E-1 |
| 0.05 | 0.390427 | 1.10E-1 | 2.74 | 1.25E-1 |
| 0.025 | 0.469776 | 3.02E-2 | 3.63 | 3.12E-2 |
| 0.0125 | 0.492262 | 7.74E-3 | 3.91 | 7.80E-3 |
| 0.00625 | 0.498054 | 1.95E-3 | 3.98 | 1.95E-3 |

## Problem 5.4 - 9

a) $f(x) = e^x$ at $x = 0$
Using the follwoing equations:
$D_h^{(2)} f(t) = \frac{f(t+h)-2f(t)+f(t-h)}{h^2}$
$\acute{f}(t) - \frac{f(t+h)-2f(t)+f(t-h)}{h^2} \approx \frac{-h^2}{12} f^(4)(t)$
We get the following results:

| $h$ | $D_h^{(2)} f(x)$ | Error | Ratio |
|---|---|---|---|
| 0.5 | 1.0210077 | -2.10E-2 | |
| 0.25 | 1.0052192 | -5.22E-3 | 4.03 |
| 0.125 | 1.0013028 | -1.30E-3 | 4.01 |
| 0.0625 | 1.0003256 | -3.26E-4 | 4.00 |
| 0.03125 | 1.0000814 | -8.14E-5 | 4.00 |

c) Similar to part a when $f(x) = tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$

| $h$ | $D_h^{(2)} f(x)$ | Error | Ratio |
|---|---|---|---|
| 0.5 | 5.9755 | 93.5 | |
| 0.25 | 20.7416 | 78.8 | 1.19 |
| 0.125 | 52.8166 | 46.7 | 1.69 |
| 0.0625 | 82.6040 | 16.9 | 2.76 |
| 0.03125 | 94.8241 | 4.68 | 3.61 |

## Problem 5.4 - 13

a) $f(x) = cosx$ at $t = \frac{\pi}{6}$

The results are shown in the following table:

| $h$ | $D_h^{(2)} f(t)$ | Error |
|---|---|---|
| 0.5 | -0.848133 | -1.789E-2 |
| 0.25 | -0.861524 | -4.502E-3 |
| 0.125 | -0.864895 | -1.131E-3 |
| 0.0625 | -0.865738 | -2.875E-4 |
| 0.03125 | -0.865967 | -5.859E-5 |
| 0.015625 | -0.865967 | -5.859E-5 |
| 0.0078125 | -0.866211 | -1.855E-4 |
| 0.00390625 | -0.867188 | -1.162E-3 |
| 0.001953125 | -0.859375 | -6.650E-3 |

We can see that the roundoff error effects are beginning to appear after $h = 0.015625$

**EXTRA QUADRATURE PROBLEMS: SOLUTIONS PROVIDED SEPARATELY**

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